If only the small disturbances about the steady-state equilibrium condition are considered, then

$$\alpha = \alpha_{ss} + \delta\alpha \quad (4a)$$

$$\beta = \delta\beta \quad (4b)$$

$$\varphi = \delta\varphi \quad (4c)$$

Linearizing Eqs. (1–3) about the equilibrium condition α_{ss} yields

$$(M_y/I_y)_D = 3\dot{\theta}^2 R_y \cos\alpha_{ss} \sin\alpha_{ss} \quad (5a)$$

$$\delta\ddot{\alpha} + 3\dot{\theta}^2 R_y\, \delta\alpha(\cos^2\alpha_{ss} - \sin^2\alpha_{ss}) = 0 \quad (5b)$$

$$\delta\ddot{\beta} - \delta\beta(R_z\dot{\theta}^2 + 3R_z\dot{\theta}^2\cos^2\alpha_{ss}) - \delta\dot{\varphi}(1 + R_z)\dot{\theta} + 3\delta\varphi R_z\, \dot{\theta}^2 \sin\alpha_{ss}\cos\alpha_{ss} = 0 \quad (5c)$$

$$\delta\ddot{\varphi} + \delta\varphi(R_x\dot{\theta}^2 + 3R_x\dot{\theta}^2\sin^2\alpha_{ss}) + \delta\dot{\beta}(1 - R_x)\dot{\theta} - 3\delta\beta R_x\dot{\theta}^2 \sin\alpha_{ss}\cos\alpha_{ss} = 0 \quad (5d)$$

In the absence of a disturbing torque, R_x and R_y are normally positive and R_z negative for stable behavior.† Thus

$$I_y \geq I_z > I_x \quad (6)$$

When a disturbing torque is present, the pitch response essentially is unaltered if α_{ss} is small. However, Eqs. (5c) and (5d) now have a $\delta\varphi$ and a $\delta\beta$ term, respectively, if α_{ss} is different from zero. The characteristic equation for the coupled roll-yaw motion is

$$\lambda^4 + \lambda^2\dot{\theta}^2(1 + 3R_x\sin^2\alpha_{ss} - 3R_z\cos^2\alpha_{ss} - R_zR_x) - 3\lambda\dot{\theta}^3\sin\alpha_{ss}\cos\alpha_{ss}(R_x + R_z) - 4R_xR_z\dot{\theta}^4 = 0 \quad (7)$$

If α_{ss} is small, the four roots are, approximately,

$$\lambda_1 = \dot{\theta}[-a + \tfrac{1}{2}(b)^{1/2}]^{1,2} + [1.5\dot{\theta}(R_x + R_z)/(b)^{1/2}]\sin\alpha_{ss} \quad (8a)$$

$$\lambda_2 = -\dot{\theta}[-a + \tfrac{1}{2}(b)^{1/2}]^{1/2} + [1.5\dot{\theta}(R_x + R_z)/(b)^{1/2}]\sin\alpha_{ss}$$

$$\lambda_3 = \dot{\theta}[-a - \tfrac{1}{2}(b)^{1/2}]^{1/2} - [1.5\dot{\theta}(R_x + R_z)/(b)^{1/2}]\sin\alpha_{ss} \quad (8b)$$

$$\lambda_4 = -\dot{\theta}[-a - \tfrac{1}{2}(b)^{1/2}]^{1/2} - [1.5\dot{\theta}(R_x + R_z)/(b)^{1/2}]\sin\alpha_{ss}$$

where

$$a = (\tfrac{1}{2})(1 - 3R_z - R_zR_x)$$
$$b = (1 - 3R_z - R_zR_x)^2 + 16\, R_zR_x$$

For the moment-of-inertia distribution given by Eq. (6), λ_1 and λ_2 are complex conjugates with negative real parts, whereas λ_3 and λ_4 are conjugates with positive real parts.‡ Thus the roll-yaw motion is unstable. If the sum of R_x and R_z is zero, the real parts of Eqs. (8a) and (8b) are zero, but the pitch restoring torque also is zero! Digital solutions of the nonlinear differential equations have verified this instability.

A disturbing torque that causes a steady-state value of β also is possible. In this case the three perturbation equations are coupled, and the characteristic equation is of the sixth degree. For small values of β_{ss}, the roots do not have any positive real parts and the solutions are stable. However, the digital solutions of the nonlinear equations diverged for values of β_{ss} greater than about 10°.

An examination of Eq. (3a) indicates that a constant torque about the x axis requires that at least two of the three orientation angles have steady-state values. This case has not been examined extensively, but a limited number of digital solutions were stable for small values of $(M_x)_D$.

The fact that a bias angle in pitch leads to instability in the roll-yaw motion indicates the highly nonlinear nature of the problem. A similar roll-yaw behavior is caused by the forced pitch motion due to orbital eccentricity.[5] Thus it is apparent that the principle of superposition is of limited use in stability analyses of gravity-gradient stabilized satellites.

† Another conditionally stable configuration is discussed in Ref. 2.

‡ This assumes α_{ss} to be positive.

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Entropy Perturbations in One-Dimensional Magnetohydrodynamic Flow

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IN a method developed by Germain and the author[1–3] for discussing weakly nonisentropic one-dimensional flows of an ideal compressible fluid, it was found that the addition of an entropy perturbation introduced a nonhomogeneous term in an otherwise homogeneous system of perturbation equations, and, further, that the entropy perturbation could be determined directly. Thus, the various problems considered could be solved by considering first the homogeneous system (isentropic perturbed flow) and then adding particular solutions to the complete system (nonisentropic perturbed flow). In two cases of interest, viz., an initially uniform or centered simple-wave flow, it was found that the addition of an entropy perturbation affected the sound speed but not the particle velocity, i.e., there was a *particular* solution with the particle-velocity perturbation equal to zero. A general discussion of this phenomenon, including necessary and sufficient conditions for it to occur, was given in Refs. 2 and 3.

The forementioned perturbation theory has been extended to one-dimensional hydromagnetic flow subjected to a transverse magnetic field,[4–7] and it was found that the addition of an entropy perturbation did not affect the particle velocity in an initially uniform flow. (This is a consequence of the result that the nonisentropic perturbation of an initially uniform flow must reduce to the solution of the corresponding problem in conventional gas dynamics in the limit of vanishing magnetic field.)[8] But this result *was not* obtained for an initially centered simple-wave flow.[5] It is the purpose of the present paper to derive conditions for the particle velocity to be unaffected by the addition of an en-

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tropy perturbation. A well-known consequence of the assumption of infinite electrical conductivity is that the ratio of magnetic induction and density is constant along each particle path, and, for a constant-state or simple-wave flow, this ratio is constant throughout the flow. Doubtless, there exist other flows for which this ratio is constant, and it is with these flows that the present paper is concerned. Then, the results for the nonmagnetic case are contained as a special case. Thus, it is assumed tacitly herein that *the density and magnetic induction are constantly proportional.*

The one-dimensional unsteady motion of an ideal, inviscid, perfectly conducting, compressible fluid, subjected to a transverse magnetic field, i.e., the induction $\mathbf{B} = (0,0,B)$, is governed by the system of equations[9]

$$P = \exp[(s - s^*)/c_v]\rho^\gamma \tag{1}$$

$$\omega_t + u\omega_x + [\omega^2 + (\gamma - 2)c^2]u_x/2\omega = 0 \tag{2}$$

$$u_t + uu_x + 2\omega^3\omega_x/[\omega^2 + (\gamma - 2)c^2] = c^2 s_x/\gamma(\gamma - 1)c_v \tag{3}$$

$$s_t + us_x = 0 \tag{4}$$

where, u, c, s, s^*, P, ρ, $b^2 = B^2/\mu\rho$, μ, γ, and $\omega = [b^2 + c^2]^{1/2}$ are, respectively, the particle velocity, local speed of sound, specific entropy, specific entropy at some reference state, pressure, density, square of the Alfvén speed, permeability, ratio of specific heat at constant pressure c_p and at constant volume c_v, and the true speed of sound, the limiting case of a fast wave.

When the system of Eqs. (1–4) is linearized in the neighborhood of a known isentropic flow, denoted by the subscript zero, a system of linear equations for the terms of first order, denoted by the subscript one, is obtained. For the problem under consideration, it is equivalent to look for solutions of this linear system with $u_1 = 0$. This gives

$$\omega_{1t} + u_0\omega_{1x} + [\omega_0^2 + (\gamma - 2)c_0^2]\omega_1 u_{0x}/2\omega_0^2 = 0 \tag{5}$$

$$(\omega_0\omega_1)_x = \frac{[\omega_0^2 + (\gamma - 2)c_0^2]}{(\gamma - 1)\omega_0^2}\,\frac{\rho_0 c_0^2\,\Omega'(\psi_0)}{2\gamma c_v} \tag{6}$$

$$d\psi_0 = \rho_0[dx - u_0 dt]$$

where $\psi_0 =$ const defines the particle paths. Thus, the problem is reduced to expressing the compatibility of Eqs. (5) and (6), and the conditions on the functions $u_0(x,t)$, $c_0(x,t)$, and $\omega_0(x,t)$ which allow this compatibility to be determined.

The characteristics of Eq. (5) are

$$\frac{dt}{1} = \frac{dx}{u_0} = \frac{d\omega_1}{-\{[\omega_0^2 + (\gamma - 2)c_0^2]/2\omega_0^2\}u_{0x}\omega_1} \tag{7}$$

One first integral of Eq. (7) is $\psi_0 =$ const. Since $\omega_1 = \omega_0$ is a particular solution of (5), the following method of solution is suggested. From Eq. (7),

$$\frac{\omega_{0t}\,dt}{\omega_{0t}} = \frac{\omega_{0x}\,dx}{u_0\omega_{0x}} = \frac{\omega_{0t}\,dt + \omega_{0x}\,dx}{\omega_{0t} + u_0\omega_{0x}} = \frac{d\omega_0}{-\{[\omega_0^2 + (\gamma - 2)c_0^2]/2\omega_0\}u_{0x}} = \frac{d\omega_1}{-\{[\omega_0^2 + (\gamma - 2)c_0^2]/2\omega_0^2\}u_{0x}\omega_1}$$

which gives the first integral $\omega_1/\omega_0 =$ const. Thus, the solution of Eq. (5) may be written as

$$\omega_1/\omega_0 = F(\psi_0) \tag{8}$$

where F is an arbitrary function.

Since Eq. (5) is obtained from the continuity equation, which is the same in the magnetic or nonmagnetic case, it follows that[3]

$$c_1/c_0 = g(\psi_0) \tag{9}$$

where g is an arbitrary function.

From Eqs. (8) and (9), it follows that

$$\frac{\omega_1}{c_1} = \frac{\omega_0}{c_0} \frac{F(\psi_0)}{g(\psi_0)} = \frac{\omega_0[\omega_0^2 + (\gamma - 2)c_0^2]}{c_0(\gamma - 1)\omega_0^2} \tag{10}$$

Thus, Eq. (6) may be written as

$$(\omega_0\omega_1)_x = \rho_0 c_0^2\left[\frac{\Omega'(\psi_0)}{2\gamma c_v}\,\frac{F(\Omega_0)}{g(\psi_0)}\right] \tag{11}$$

Substituting (8) into (11) gives

$$\frac{\omega_{0x}}{\rho_0\omega_0} = \frac{c_0^2}{\omega_0^2}\left[\frac{\Omega'(\psi_0)}{4\gamma c_v g}\right] - \frac{F'(\psi_0)}{2F} \tag{12}$$

Writing $\rho_0 = \text{cst}c_0^{2/(\gamma-1)}$ and absorbing the constant in the arbitrary functions, replacing $\omega_{0x}/\omega_0 = c_{0x}\,F(\psi_0)/c_0 g(\psi_0)$, and taking the material derivative in Eq. (12), the following condition is obtained:

$$\frac{D}{Dt}\left[\frac{c_{0x}}{c_0^{(\gamma+1)/(\gamma-1)}}\right] = \frac{\Omega'(\psi_0)}{4\gamma c_v F}\,\frac{D}{Dt}\left[\frac{c_0^2}{\omega_0^2}\right] \tag{13}$$

Carrying out the indicated differentiation and replacing the derivative c_{0xt} by its equivalent obtained by differentiating the continuity equation with respect to x, there results the condition

$$\frac{u_{0xx}}{c_0^{2/(\gamma-1)}} = -\frac{\Omega'(\psi_0)}{2(\gamma - 1)\gamma c_v F}\,\frac{D}{Dt}\left[\frac{c_0^2}{\omega_0^2}\right] \tag{14}$$

For the nonmagnetic case, the right-hand side vanishes so that the necessary and sufficient condition for the existence of a particular solution $u_1 = 0$ is $u_{0xx} = 0$, i.e., $u_0(x,t)$ is a linear function of x.

By carrying out the indicated differentiation in Eq. (14), this condition may be written as

$$\frac{D}{Dt}\left[\frac{u_{0xx}}{u_{0x}}\,\frac{\omega_0^2}{c_0^2 c_0^{2/(\gamma-1)}}\right] = 0 \tag{15}$$

Equation (15) may be put into the following form:

$$\frac{D}{Dt}\left[\frac{u_{0xx}}{u_{0x}}\right] + \left[1 + (\gamma - 2)\,\frac{(\omega_0^2 - c_0^2)}{\omega_0^2}\right]u_{0xx} = 0 \tag{16}$$

Thus, one has the final result: the necessary and sufficient condition for the existence of a particular solution $u_1 = 0$ in the magnetic case is that (u_0, c_0, ω_0) satisfy Eq. (16). It should be noted that $u_{0xx} = 0$ is a *sufficient* condition.

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Inversion Property of the Fundamental Matrix in Trajectory Perturbation Problems

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Systems described by ordinary linear differential equations with time varying coefficients may be analyzed conveniently using the concepts of state variables and fundamental matrix. Characteristically, the inverse of this matrix appears in the state transition equation. An inversion property of the fundamental matrix applicable to a class of dynamic systems which includes as a member trajectory perturbation problems is presented. This property allows the inverse matrix to be obtained by a simple rearrangement of elements of the original matrix. When the matrix is of high order, significant advantages accrue in both time saving and numerical accuracy.

INCREASED emphasis has been given recently to the application of linear perturbation techniques in studies of trajectory and guidance problems.[1–3] The resulting perturbed equations of motion are given by a set of ordinary linear differential equations with time-varying coefficients. The solution of such a set can be facilitated greatly by the concepts of "state variables" and "fundamental matrices," where the state transition equations are expressed in terms of these computable matrices. Characteristically, the inverse of the fundamental matrix appears in the equations. It is recognized that inversion of high-order matrices can be both time consuming and inaccurate even with the aid of digital computers. Fortunately, in the case of perturbed trajectories there exists an inversion property, which allows the inverse to be obtained by a simple rearrangement of elements of the origin matrix. Such a property has been indicated by McLean et al.[2] for the special case of coasting trajectories. The purpose of the present paper is to extend the inversion property to a class of dynamic systems which includes as a member trajectory motion influenced by an acceleration forcing function (e.g., thrust acceleration) in addition to gravitational acceleration. Also, it is felt that the usefulness of the inversion property deserves wider attention.

State Equations and Fundamental Matrix

Consider a linear system described by a set of n first-order differential equations. In vector and matrix notation

$$(ds/dt) - A(t)s(t) = B(t)f(t) \tag{1}$$

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where s is an n-dimensional state vector, f is an m-dimensional vector of forcing inputs applied to the system, and A and B are $n \times n$ and $n \times m$ coefficient matrices, respectively. The state is defined as a set of output variables from which the entire future behavior of the system may be determined, provided the future inputs to the system are known. Assume initialization of the problem at a fixed time t_0 with corresponding state $s(t_0)$. In general, two types of problems are admitted: one where the region of interest lies between fixed-time interval (t_0, t_f), and the other where a terminal t_f is not specified. In either case the solution of Eq. (1) may be facilitated by introducing an $n \times n$ fundamental matrix $\Lambda(t)$, which satisfies the following equation:

$$(d\Lambda/dt) + \Lambda(t)A(t) = 0 \tag{2}$$

and is subject to an arbitrary boundary condition to be discussed presently. In the literature, Eq. (2) often has been called the adjoint equation to Eq. (1), and Λ is the adjoint matrix.

Premultiplying Eq. (1) by Λ, postmultiplying Eq. (2) by s, and adding the two modified equations yields

$$(d/dt)(\Lambda s) = \Lambda(t)B(t)f(t)$$

When this equation is integrated between the limits t_1 and t_2, the general state transition equation is

$$s(t_2) = \Lambda^{-1}(t_2)\Lambda(t_1)s(t_1) + \Lambda^{-1}(t_2) \int_{t_1}^{t_2} \Lambda(t)B(t)f(t)dt \tag{3}$$

Nonsingularity of Λ is assumed, and the superscript -1 denotes the matrix inverse operation. Several interpretations of this equation are as follows:

1) Suppose the problem-definition does not specify a fixed terminal time. A convenient choice of boundary condition for Eq. (2) is $\Lambda(t_0) = I$ (identity matrix). Letting $t_1 = t_0$ and $t_2 = t$, Eq. (3) gives the general solution for $s(t)$ in terms of the initial state and the effect of $f(t)$ over the interval (t_0, t). If A, B, and f are assumed to be known functions of time, Eq. (1) does not have to be solved repeatedly for different values of the initial state.

2) Suppose a fixed terminal time t_f is specified and the terminal state is of primary interest. A convenient choice of boundary condition is $\Lambda(t_f) = I$, and $\Lambda(t)$ is computed by integrating Eq. (2) backwards in time. Letting $t_2 = t_f$ and $t_1 = t$, Eq. (3) gives the terminal state in terms of the instantaneous state and the effect of $f(t)$ over (t, t_f). If a desired terminal state is specified and $s(t)$ is measured, then synthesis of a control function $f(t)$ may proceed from the terminal form of Eq. (3).

3) Consider a dynamic process that is to be controlled repetitively based on sampled measurements of the time-varying state. Assume that the measurements are contaminated by random noise, and assume that a statistical filtering and prediction procedure is employed to improve the state measurements. The deterministic prediction equation is given by Eq. (3) and may be operated on statistically.

The previous development indicates the requirement for inverting the fundamental matrix. An inversion property, which allows great simplification of this operation, is presented now for a special class of systems.

Inversion Property of the Fundamental Matrix

Consider a class of systems having the following restrictions: 1) the number of state variables is even; and 2) the system coefficient matrix A can be partitioned into four square submatrices, each of order $n/2$, such that the diagonal submatrices are equal to the null matrix and the off-diagonal submatrices are symmetrical.

A common example of an even-ordered state vector is a set of output variables and their first derivatives. If a system